CHAPTER – 4

THEORY OF PROBABILITY

Introduction

The theory of probability is the science of providing a numerical measure to the uncertainty that prevails in various events that we encounter in our everyday life. Probability may be defined as the science that deals with uncertainties and helps us to take decisions about actions even in the midst of uncertainties. Probability theory had its origin in the games of chance. Gamblers used it earlier, to find the most probable case, in case of different games of chance. Two French mathematicians, Blice Pascal and P. Frermat, laid the foundation of mathematical theory of probability in the mid-seventeenth century. After that scientists like James Bernoulli (Treatise of Probability), De-Moivre (Doctrine of Chances), Thomas Bayes (Inverse Probability), P.S. Laplace (Theoric Analytique Des Probabilities) made out standing contributions in the field of probability. Another group of mathematicians from Russia like Chebychev (Chebychev's Inequality), Liapounoff (Central Limit theorem) and Khintchine (Law of Large Numbers) made noteworthy offering towards the theory of probability.

The theory of probability was historically used to simplify the complexities of betting and games. In course of time, the probability theory was incorporated in business process and decision making apparatus by business firms, government, professional and non–professional organizations.

Probability theory was used in economic decision–making, business forecasting, weather forecasting, industry etc. The theory can be gainfully applied when decision has to be taken in the midst of uncertainties and there is the existence of risk involved with every decision taken. Predictions on demand for a new product, estimation of the cost of production, chance of failure in a project, chance of insurance claims, forecasting crop failure in agriculture etc. can be better understood and explained with the help of the concept of probability.

Basic Terminology

Random Experiment: It is an experiment, which if conducted repeatedly under homogeneous conditions does not give the same result. One may know the set of all possible outcomes that the experiment will result but the exact result cannot be predicted with certainty. For example, when a coin is tossed we get either head or tail, i.e., there are two possibilities out of which any one of the outcomes may occur (i.e., either Head or Tail).

Trial and Event: Performing of a random experiment is called as a trial and the result of the random experiment thus performed is called as the event. For example, tossing of a coin is a trial and getting either head or tail is an event.

Event is called Simple or Elementary, if it corresponds to a single possible outcome of the trial, otherwise it is compound or composite event. An event whether simple and composite is generally denoted by capital Latin letters like, A, B, E etc. Thus, in throwing a single die, the event of getting 5 is a simple event but the event of getting an odd number is a composite event as it consists of three elementary points 1, 3 and 5.

Exhaustive Events or Cases: The collection of all possible outcomes of a random experiment is called exhaustive cases. In the toss of a single coin we get either a head or a tail. Hence, there are two exhaustive cases. If two coins are tossed exhaustive cases will be $2^2 = 4$. If two dice are thrown then the exhaustive cases will be $6^2 = 36$.

Favourable Cases: The number of outcomes of a random experiment which result in the happening of a desired event are called as favourable cases. For example: In throwing a die the cases favourable to event of getting an even number is 3, and they are 2, 4 and 6. For example, in drawing a card from a pack of cards the cases favourable to getting an spade is 13, and to getting an ace is 4 and a king of diamond is 1.

Mutually Exclusive Events or Cases: Events are said to be mutually exclusive if the occurrence of one prevents the occurrence of all others. It means that if one of them occurs, it is certain that other events will not occur. Mutually exclusive events are also called as disjoint events. For example, in tossing of a coin head and tail are mutually exclusive events i.e. if head occurs tail will not occur and vice versa.

Equally Likely Cases: The outcomes are said to be equally likely if there is no reason to prefer one rather than the other. For example, in tossing of a coin, all outcomes (H, T) and in rolling a dice all outcomes (1,2,3,4,5,6) are equally likely if the coin or the dice are unbiased.

Independent Events: Two events are said to be independent of each other if the occurrence or non–occurrence of one of them does not affect the occurrence or non–occurrence of the other. For example, in the toss of a coin repeatedly getting head in the first throw is independent of getting head in the second or in any of the of the subsequent throws. Again, if a coin is tossed and a die is thrown together, then the result of the toss is independent of the face value of the die.

Sample Point and Sample Space: A sample point is an elementary event. The set of all possible outcomes i.e. the collection of all possible sample points of a random experiment is called a sample space. For example in throwing of a die there are six sample points. All these sample points taken together forms the sample space. The sample space is generally denoted by 'S' or by 'Ω' and a sample point is represented by 's'.
In this case we have, S = {1,2,3,4,5}.

Complementary Event: Two events A and B, are said to be complementary to each other, if the event B represents the non occurrence of the event A. The event complementary to the event A is generally denoted by A^c, A' or A. The complementary event A', of the event A consists of all the sample points of the sample space that are not included in the event A. Thus we have,

$$P(A) + P(A') = 1$$

For example, if A is the event of getting a multiple of 3 when a die is rolled. Then, we have A = { 3, 6}and A' = {1, 2, 4, 5 }.

Union of Two Events: The union of two events, generally denoted by $A \cup B$ represents those sample points which belongs either to A or to B or to both. The union may be extended to any number of events.

Intersection of Two Events: The intersection between the two events, generally denoted by $A \cap B$ represents those sample points which belongs to both A and B. The intersection between n events consists of those sample points that is common to all the n events.

Conditional Probability: Sometimes it so happens that the probability of an event A(say) depends on the occurrence or non-occurrence of another event B. For example, the probability of a person carrying an umbrella is high if the day is cloudy or rainy and less otherwise. Thus, if A is the event that the person carries an umbrella and B be the event that the day is cloudy. Then the event A is dependent on B. The symbol A|B is the event that the man carries an umbrella when it is known that the day is cloudy. However, if A does not depend on B then the event A|B is equivalent to the ordinary event A.

Classical or Mathematical Definition of Probability

If there are n, mutually exclusive, equally likely and exhaustive cases out of which m of them are favourable to an event A, then

$$P(A) = \frac{m}{n} = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}}$$

Properties of Probability

1. Probability of an event lies between 0 and 1.That is if A is an event, then $0 \leq P(A) \leq 1$.
2. Probability of an impossible event is 0.
3. Probability of a certain event is equal to 1.
4. If the occurrence of an event A implies the occurrence of another event B then $P(A) \leq P(B)$.
5. Probability of happening of a particular event and the probability of non–happening of that event when added results to unity i.e., $P(A) + P(\bar{A}) = 1$

Limitations of the Classical Approach to Probability

1. It is based on the feasibility of subdividing the possible outcomes of the experiment into 'mutually exclusive', 'equally likely' and 'exhaustive cases'.
2. It fails when n, total number of outcomes, of the random experiment is infinite or actual value of n is not known.
3. The classical approach has limited applications, as in most cases it may not be possible to enumerate all the outcomes of a random experiment.
4. 'Equally likely' is synonymous to equally probable. Thus, the theory fails to compute the probability if the outcomes of a random experiment are not equally likely. In some cases it may be difficult to know whether outcomes of a random experiment are equally likely or not.

Statistical Definition of Probability

This definition is also termed as the frequency approach to probability. Due to certain limitations of the classical definition of probability, Von Moses has developed an alternative definition of probability. In this definition, probability of an event is considered as relative frequency of an event. Suppose we perform a sequence of n repetitions of an experiment. Let m be the number of times, out of these n repetitions in which a particular event A (say) occurs.

Then $\frac{m}{n}$ is called as the relative frequency of the event A. If many repetitions of n trials are made and if each time the value of relative frequency is recorded then we would find that the relative frequencies differ from one another by small amounts only, provided n is very large.

In such cases there is a tendency in the relative frequencies to accumulate in the neighbourhood of some fixed value. This limiting value as $n \to \infty$ is regarded as the probability of E in the experiment. The basis of this definition is the concept of statistical regularity.

Advantages

The frequency definition has two important advantages over the classical definition.

(i) This definition can be applied even if the number of trials is infinitely large.

(ii) In this definition all the elementary events need not be equally likely.

Disadvantages

With the help of this definition the probability of an event, cannot be exactly determined. It is determined empirically. There is also no guarantee that the limit of relative frequencies will exist. From the mathematical point of view this frequency definition has been discarded and an alternative definition has been developed which is based on some axioms.

Axiomatic Definition of Probability

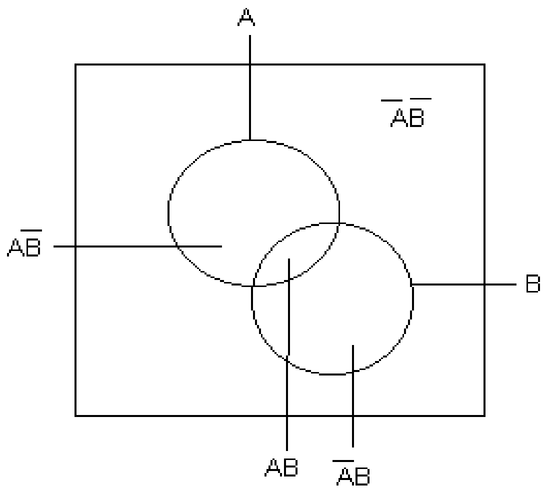
The axiomatic definition of probability is based on the following axioms. Let '*S*' be the sample space and 'A' be any event defined in S. Then *P( )* is called the probability function, defined on the sample space *S* provided the following axioms are satisfied:

1. Let $P(A)$ is the probability of the event A. Then $P(A)$ is a real number such that $P(A) \geq 0$, for every event A in S.
2. Total probability $P(S) = 1$
3. If $A_1, A_2, \ldots, A_n$ be small n mutually exclusive events, then $P(A_1+A_2+\ldots+A_n) = P(A_1) + P(A_2)+\ldots+P(A_n)$

Comparison between Classical Approach and Axiomatic Approach

1. The classical theory is easier to understand and is based on arithmetic considerations while to understand the axiomatic approach one has to be conversant with the properties of set theory.
2. In the classical theory, all theorems and results are obtained on the basis of logical arguments. In the axiomatic theory, all results are based on the axioms by using the mathematical properties of sets.
3. The classical theory is based upon the concept of "equally likely cases" when the number of possible outcomes is only finite. The axiomatic tic theory is quite general and embraces all cases whether equally likely or not and irrespective of whether the number of possible members is finite or infinite.
4. The event in case of axiomatic approach is the subset of the universal set the sample space. But, in case of the classical approach the event is a phenomenon, which arises from the performance of a random experiment.
5. In the classical theory, "probability" is defined as a ratio or relative frequency of two positive whole numbers. In the axiomatic theory, "probability" is simply a non-negative number attached with the event, i.e. probability is a set-function obeying the axioms.
6. The addition theorem of probability is based on the classical theory but in the axiomatic theory this theorem is not derived but is a rule by Axiom 3.

General Additive Theorem



The occurrence of atleast one of the two events A and B, when A and B are independent events are given by

$$P(A+B) = P(A) + P(B) - P(AB)$$

Proof: Let A and B be two independent events.

From the diagram it is clear that the event A can be decomposed into two parts –

$A = (A-AB) + AB$

$\therefore P(A) = P(A-AB) + P(AB)$ (i)

Similarly,

$B = (B-AB) + AB$

$\therefore P(B) = P(B-AB) + P(AB)$

$\Rightarrow P(B-AB) = P(B) - P(AB)$ (ii)

Again, $(A+B)$ can be decomposed into three parts –

$A + B = (A-AB) + (AB) + (B-AB)$

$\Rightarrow P(A + B) = P(A-AB) + P(AB) + P(B-AB)$

$= P(A) - P(AB) + P(AB) + P(B) - P(AB)$

[using (i) and (ii)]

$= P(A) + P(B) - P(AB)$

Corollary:

For three independent events namely, A, B and C

$P(A+B+C) = P(A+B) + P(C) - P[(A+B) C]$

$= P(A) + P(B) - P(AB) + P(C) - P(AC+BC)$

$= P(A) + P(B) + P(C) - P(AB) - [P(AC) + P(BC) - P(ABC)]$

$= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

Two Basic Laws of Probability

1. Additive Law of Probability or The Theorem of Total Probability

The probability that either of the two mutually exclusive events A and B will occur is given by the sum of their individual probabilities. In symbols we may write

$$P(A + B) = P(A) + P(B)$$

Proof: Suppose that the total number of mutually exclusive, equally likely and exhaustive cases be n. Out of which m_1 and m_2 of them be favourable to the events A and B respectively. Thus, by the classical definition of probability we have

$$P(A) = \frac{m_1}{n} \text{ and } P(B) = \frac{m_2}{n}$$

Again the number of cases that are favourable either to A or to B is $m_1 + m_2$. So, we have,

$$P(A+B) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$$

✍ **Note**

The theorem can be generalized for n mutually exclusive events. If $A_1, A_2, \ldots, A_n$ be n mutually exclusive events then

$P(A_1 + A_2 + \ldots + A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Multiplicative Law of Probability or Theorem of Compound Probability

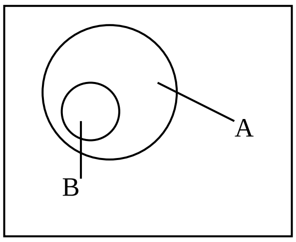
The probability of the simultaneous occurrence of two events A and B is equal to the probability of A multiplied by the conditional probability of B given that A has already occurred. (Or it is equal to the probability of B multiplied by the conditional probability of A given that B has already occurred.)

In symbols, $P(AB) = P(A)\,P(B|A)$

$= P(B)\,P(A|B)$

Proof: Let n denotes the total number of equally likely cases among which m cases are favourable to the event A. The cases favourable to both A and B are included in the m cases that are favourable to A. Let that number be m_1. Then we have from the classical definition of probability-

$$P(AB) = \frac{m_1}{n} = \frac{m}{n} \times \frac{m_1}{m} = P(A) \times \frac{m_1}{m}$$



Since m_1/n represents the conditional probability of B when A has already occurred. So, $P(B|A) = m_1/n$. This is because if we assuming that A has already occurred then we have only m equally likely cases left, out of which m_1 are favourable to B.

Thus we have, $P(AB) = P(B)\,P(B|A)$

Since the compound event AB involves A and B symmetrically so we can have

$P(AB) = P(B)\,P(A|B)$

Corollary: If A and B are independent events P(B|A) is same as P(B). then for two independent events the theorem of compound probability becomes

$P(AB) = P(A) \times P(B)$

✍ **Note**

The relation P(AB) = P(B) P(A/B) is often used to find the conditional probability of the event P(A/B). The relation is explained by cross multiplication $P(A|B) = P(AB) / P(B)$ provided $P(B) \neq 0$. Similarly, P(B/A) can also be obtained.

Formulae

1. $P(A) = \frac{m}{n}$
2. $0 \le P(A) \le 1$
3. If $A \subseteq B$, then $P(A) \le P(B)$
4. $P(A) + P(\bar{A}) = 1$
5. $P(A+B) = P(A) + P(B) - P(AB)$
6. For mutually exclusive events A and B we have, $P(A+B) = P(A) + P(B)$
7. $P(AB) = P(A)\,P(B|A) = P(B)\,P(A|B)$
8. If A and B are independent events then $P(AB) = P(A) \times P(B)$

SOLVED ILLUSTRATIONS

BASED ON CLASSICAL APPROACH

Illustration 1: From a pack of 52 cards two cards are drawn at random. Find the probability that one is a king and the other a queen.

Solution: In a pack there are 52 cards. Two cards can be drawn in ${}^{52}C_2$ ways, which is the total number of cases.

Again, there are four kings and four queens of four different suits. A king can be drawn out of 4 kings in ${}^4C_1$ ways. Similarly, a queen can be drawn out of 4 queens in ${}^4C_1$ ways. Simultaneous occurrence of both the events will follow the multiplicative law giving the number of favourable cases as ${}^4C_1 \times {}^4C_1$ ways.

$$\therefore \text{Required probability} = \frac{{}^4C_1 \times {}^4C_1}{{}^{52}C_2} = \frac{4 \times 4}{\frac{(52)!}{(52-2)!\,(2)!}} = \frac{16}{\frac{52 \times 51 \times (50)!}{(50)! \times 2}}$$

$$= \frac{16 \times 2}{52 \times 51} = \frac{8}{663}$$

Illustration 2: A card is drawn at random from a standard pack of 52 playing cards; what is the probability that the card is

(i) either a king or a queen;
(ii) either a red card or an ace?

Solution: In a pack of playing cards there are 52 cards in all. So, one card can be drawn in 52 ways, which is the exhaustive number of cases.

(i) Let A be the event that a king occurs and B be the event that a queen occurs. We have 4 kings and 4 queens in a full pack of 52 cards. Thus $P(A) = 4/52$ and $P(B) = 4/52$. So, the probability that either a king or a queen occurs is given by P(A+B). But A and B are mutually exclusive events. So, we have

$$P(A+B) = P(A) + P(B) = 4/52 + 4/52 = 8/52 = 2/13$$

(ii) Now, let A be the event that a red card occurs and B be the event that an ace card occurs. We have 26 red cards and 4 aces.
Thus $P(A) = 26/52$ and $P(B) = 4/52$. Now, out of the four aces two are red. Thus occurrence of an ace card of red color leads to the occurrence of both A and B. So, we have $P(AB) = 2/52$ (since there are two ace cards that are red in color).

Thus, $P(A+B) = P(A) + P(B) - P(AB) = 26/52 + 4/52 - 2/52 = \frac{26 + 4 - 2}{52} = 7/13$

Illustration 3: What is the probability that a leap year selected at random will contain 53 Mondays?

Solution: We know that a leap year consists of 366 days, i.e., 52 complete weeks and two extra days. These two extra days may be any one of the following cases:
(i) Sunday and Monday (ii) Monday and Tuesday (iii) Tuesday and Wednesday (iv) Wednesday and Thursday (v) Thursday and Friday (vi) Friday and Saturday (vii) Saturday and Sunday
Out of the above seven exhaustive cases (i) and (ii) consists of Monday, which are the favourable cases.

Therefore, the required probability = $2/7$

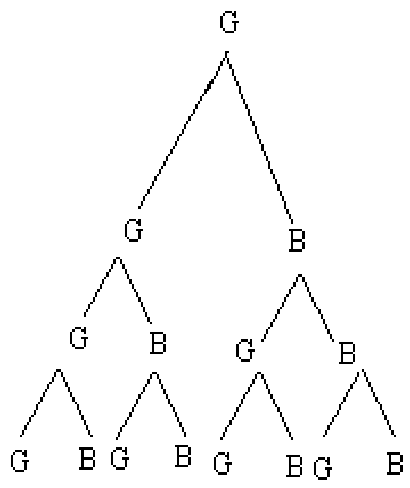
Illustration 4: Two dice are thrown. Find the probability that the sum of the points in the two dice is equal to 7.
Solution: In order to solve this problem we first draw the sample space of all possible outcomes that may occur when two dice are thrown:

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Thus, we see that the total number of outcomes when two dice are thrown is 36. Now, we find that out of these 36 cases, there are only 6 case sums up to 7. The cases are (1,6) (2,5) (3,4) (4,3) (5,2) (6,1). Thus, required probability is 6/36 = 1/6

Illustration 5: In a family there are four children. Find the probability that (i) there is exactly two girls. (ii) there is atmost two girls (iii) there is atleast two girls. Assuming the birth of a boy and girl are equally likely.
[Though this problem can be solved using Binomial distribution but here we use tree diagram which is a much simpler method.]

Solution: Let us draw the tree diagram assuming that the first child is a girl.
Here G represents a girl and B represents a boy.
From the tree diagram we can find the sample space. The sample space S is given by:
S = {GGGG, GGGB, GGBG, GGBB, GBGG, GBGB, GBBG, GBBB, BGGG, BGGB, BGBG, BGBB, BBGG, BBGB, BBBG, BBBB }



Thus, we see that the total number of outcomes of the random experiment is equal to 16.

(i) Exactly two girls
Now if we look at the sample space S we find that there are 6 outcomes in which there are exactly two girls namely { GGBB, GBGB, GBBG, BGGB, BGBG, BBGG}.
So, P [Exactly 2 girls in a family of 4] = 6/16 =3/8
(ii) Atmost two girls
Almost two girls means those cases in which there are either no girl or one girl or two girls. If we look at the sample space S we find that there are 11 outcomes in which there are exactly two girls namely { GGBB, GBGB, GBBG, BGGB, BGBG, BBGG, GBBB, BBGB, BBBG, BGBB, BBBB}.
So, P [Atmost 2 girls in a family of 4] = 11/16
ii) Atleast two girls

Atleast two girls means those cases in which there are either 2 girls or 3 girls or 4 girls. If we look at the sample space S we find that there are 11 outcomes in which there are exactly two girls namely { GGBB, GBGB, GBBG, BGGB, BGBG, BBGG, BGGG, GGBG, GGGB, GBGG, GGGG}.
So, P [Atleast 2 girls in a family of 4] = 11/16

BASED ON COMPLEMENTARY AND INDEPENDENT EVENTS

Illustration 6: A problem of Statistics is given to three students Amar, Akbar and Anthony whose chances of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that (a) the problem will not be solved and (b) the problem will be solved?

Solution: The probability that Amar will solve the problem = $\frac{1}{2}$
So, the probability that Amar will not solve the problem $=1-\frac{1}{2} = \frac{1}{2}$
The probability that Akbar will solve the problem = $\frac{1}{3}$
So, the probability that Akbar will not solve the problem $=1-\frac{1}{3}=\frac{2}{3}$
The probability that Anthony will solve the problem = $\frac{1}{4}$
So, the probability that Anthony will not solve the problem $=1-\frac{1}{4}=\frac{3}{4}$
Now, all the students will try to solve the problem independently. Since the events are independent, so the probability that all the three students will fail to solve the problem= $\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1}{4}$
The probability that the problem will be solved $=1-\frac{1}{4}=\frac{3}{4}$

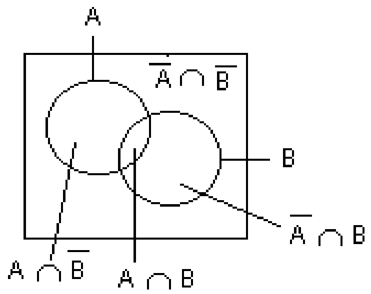
Illustration 7: The probability that a man will be alive 25 years is 3/5 and the probability that his wife will be alive 25 years is 2/3. Find the probability that –
(a) Both will be alive (b) Only the man will be alive
(c) Only the wife will be alive (d) Atleast one will be alive.

Solution: Let A be the event that the man will be alive 25 years and B be the event that his wife will be alive 25 years.
By the question, we have
P(A) = $\frac{3}{5}$ so, P(The man will die in 25 years)=P($\bar{A}$) = $1-\frac{3}{5}=\frac{2}{5}$ and
P(B) = $\frac{2}{3}$ so, P(His wife will die in 25 years)= P($\bar{B}$) = $1-\frac{2}{3}=\frac{1}{3}$
(a) P(Both will be alive) = P(AB)
= P(A) P(B) | Since, the events are independent
= $\frac{3}{5}\times\frac{2}{3}=\frac{2}{5}$
(b) P(Only man will be alive)
= P(A $\bar{B}$) = P(A) P($\bar{B}$) = $\frac{3}{5}\times\frac{1}{3}=\frac{1}{5}$
(c) P(Only the wife will be alive)
= P($\bar{A}$ B) =P($\bar{A}$) P(B) = $\frac{2}{5}\times\frac{2}{3}=\frac{4}{15}$
(d) P(Atleast one will be alive)
= 1– P(None will be alive)
= 1– P($\bar{A}$ $\bar{B}$) =1– P($\bar{A}$) P($\bar{B}$) = $1-\frac{2}{5}\times\frac{1}{3}=1-\frac{2}{15}=\frac{13}{15}$

Since, independence of A and B $\Rightarrow$ independence of
(i) A and $\bar{B}$
(ii) $\bar{A}$ and B
(iii) $\bar{A}$ and $\bar{B}$

Illustration 8. Let A and B be two independents. Then show that
i) A and $\bar{B}$ are independent events
(ii) $\bar{A}$ and B are independent events
(iii) $\bar{A}$ and $\bar{B}$ are independent events.

Solution: To prove these, let us first draw the venn diagram of two independent events A and B.



(i) From the diagram it is clear that $A \cap \bar{B} = A - A \cap B$
$\Rightarrow P(A \cap \bar{B}) = P(A) - P(A \cap B)$
$= P(A) - P(A)\,P(B) = P(A)\,(1 - P(B)) = P(A)\,P(\bar{B})$
Thus, $P(A \cap \bar{B}) = P(A)\,P(\bar{B}) \Rightarrow$ A and $\bar{B}$ are independent events.
(ii) From the diagram we find that $\bar{A} \cap B = B - A \cap B$
$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$
$= P(B) - P(A)\,P(B) = P(B)\,(1 - P(A)) = P(\bar{A})\,P(B)$
Thus, $P(\bar{A} \cap B) = P(\bar{A})\,P(B) \Rightarrow \bar{A}$ and B are independent events.
(iii) Once again, from the diagram we can find that
$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$
$= 1 - [\,P(A) + P(B) - P(A \cap B)] = 1 - P(A) - P(B) + P(A \cap B)$
$= 1 - P(A) - P(B) + P(A) \times P(B) = 1 - P(A) - P(B)(1 - P(A))$
$= (1 - P(A))\,(1 - P(B))$
$= P(\bar{A}) \times P(\bar{B})$
Thus, $P(\bar{A} \cap \bar{B}) = P(\bar{A}) \times P(\bar{B})$ which, implies that $\bar{A}$ and $\bar{B}$ independent events.

<u>BASED ON ADDITIVE AND MULTIPLICATIVE LAW</u>

Illustration 9: Out of 100 students in a boys hostel 80 take tea, 40 take coffee and 25 take both. Find the probability that a student takes (a) either tea or coffee (b) neither tea or coffee.

Solution: Let A denotes the event that the students takes tea and B denotes the event that the student takes coffee. Now by the question we have,
$P(A) = 80/100 \qquad P(B) = 40/100 \qquad P(AB) = 25/100$
The, probability that the student selected at random takes either tea or coffee is given by,
$P(A+B) = P(A) + P(B) - P(AB) = 80/100 + 40/100 - 25/100 = 95/100 = 19/20$
The, probability that the student selected at random takes neither tea nor coffee is given by
$P(\bar{A}\bar{B}) = 1 - P(A+B) = 1 - 19/20 = 1/20$

Illustration 10. If A and B are independent events with P(A) = 0.6 and P(B) = 0.2. Find P(A+B).

Solution: We know that for two independent events A and B
$P(A\,B) = P(A)\,P(B) = 0.6 \times 0.2 = 0.12$
Now, $P(A+B) = P(A) + P(B) - P(A\,B) = 0.6 + 0.2 - 0.12 = 0.68$

Illustration 11. For any two events A and B, prove that
$P(AB) \le P(A) \le P(A+B) \le P(A) + P(B)$

Solution: From the multiplicative theorem we have,
$P(AB) = P(A)\,P(B/A)$
Now, we have $P(B/A) \le 1$ from the definition of probability.
$\Rightarrow P(A) \ge P(AB)$ (1)
Again, we know that

$P(A + B) = P(A) + P(B) – P(AB)$

$= P(A) + P(\bar{A}B+AB) – P(AB)$

$= P(A) + P(\bar{A}B) + P(AB) – P(AB) = P(A) + P(□B)$

Now, $P(\bar{A}B) \geq 0$, this implies $P(A + B) \geq P(A)$ (2)

Also, $P(A + B) = P(A) + P(B) – P(AB)$

So, we see that since P(AB) is non negative, thus P(A + B) cannot exceed P(A) + P(B),

That is $P(A + B) \leq P(A) + P(B)$ (3)

Thus, combining (1), (2) and (3) we have

$P(AB) \leq P(A) \leq P(A + B) \leq P(A) + P(B)$

Illustration 12: The odds that a book will be favorably reviewed by 3 independent critics are 5:2, 4:3 and 3:4 respectively. What is the probability that out of the three reviews a majority will be preferable?

Solution: By the question we have,

The probability that the first critic will favour the book is $\frac{5}{7}$

The probability that the second critic will favour the book is $\frac{4}{7}$

The probability that the third critic will favour the book is $\frac{3}{7}$

Out of the three critics majority will favour the book means it is favoured by 2 or 3 critics.

2 critics will favour the book will have the following three mutually exclusive ways:

(I) P_1 = First and second favour the book and the third disfavour it.

$$= \frac{5}{7} \times \frac{4}{7} \times \frac{4}{7} = \frac{80}{343}$$

(II) P_2 = First and third favour the book and the second disfavour it.

$$= \frac{5}{7} \times \frac{3}{7} \times \frac{3}{7} = \frac{45}{343}$$

(III) P_3 = Second and third favour the book and the first disfavour it.

$$= \frac{2}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{24}{343}$$

Again, all the 3 critics favour the book will have the probability

$$P_4 = \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{60}{343}$$

Thus, the required probability is $= P_1 + P_2 + P_3 + P_4 = \frac{45}{343} + \frac{80}{343} + \frac{24}{343} + \frac{60}{343} = \frac{209}{343}$

✍ **Note**

If odds in favour of an event is a : b then we have, P[of the occurance of that event] = a/(a+b) and P[of non-occurrence of that event] = b/(a+b). If odds against an event is a : b then we have P[of occurrence of that event] = b/(a+b) and P[of non-occurrence of that event] = a/(a+b).

Illustration 13: Assuming that in the city the probability that a worker selected at random has a monthly income of more than Rs. 1000 is 0.625. Assume further that the probability of the worker to have a monthly income of less than Rs. 1500 is 0.765. Under these two conditions what is the probability that the monthly income of the worker selected at random lies between Rs. 1000 and Rs. 1500?

Solution: Let A be the event of getting more than Rs. 1000 and B be the event of getting less than Rs. 1500.

So, AB is the event that the monthly income of the worker selected at random lies between Rs. 1000 and Rs. 1500
By the question,
P(A) = 0.625, P(B) = 0.765
We know that the total probability is equal to 1, thus we have
P(A+B) = P(A) +P(B) – P(AB)
$\Rightarrow$ 1 = 0.625 + 0.765 – P(AB)
$\Rightarrow$ P(AB) = 1.39 – 1
$\Rightarrow$ P(AB) = 0.39

Illustration 14: A purse contains 2 silver coins and 4 copper coins and a second purse contains 4 silver coins and 3 copper coins. If a coin is selected at random from one of the two purses, what is the probability that it is a silver coin?

Solution: Let A be the event of selecting a silver coin.
B_1 be the event that the first purse is selected
B_2 are the events that the second purse be selected.
The silver coin can be selected in the following two mutually exclusive cases
(a) The first purse is selected and a silver coin is drawn from it i.e. B_1A
(b) The second purse is selected and a silver coin is drawn from it i.e. B_2A
Therefore, the required probability =
$P(B_1A + B_2A) = P(B_1A) + P(B_2A$ | As the events are mutually exclusive
$= P(B_1)\ P(A/B_1) + P(B_2)\ P(A/B_2)$

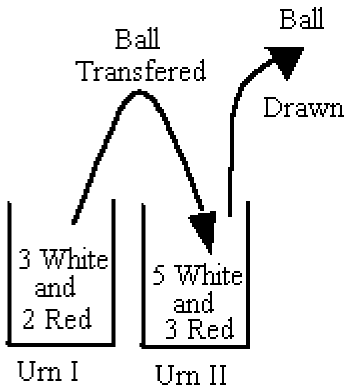
$$= \frac{1}{2} \times \frac{2}{6} + \frac{1}{2} \times \frac{4}{7} = \frac{1}{6} + \frac{2}{7} = \frac{7+12}{42} = \frac{19}{42}$$

Illustration 15: A bag contains 4 red and 6 green ball. Two draws of one ball each are made without replacement. What is the probability that one is red and the other is green?

Solution: Let A be the event that a red ball is drawn and let B be the event that a green ball is drawn.
So, P(A)= 4/10 and P(B) = 6/10
Probability of drawing a green ball in the second draw given that the first draw has given a red ball P(B/A) = 6/9 (since only 9 balls are left out of which 6 are green)
Now, Probability of the combined event A and B is given by
P(AB) = P(A) × P(B/A) = 4/10 × 6/9 = 24/90
But it could also happen the first ball may be green and second ball is red.
Probability of drawing a green first P(B) = 6/10 and red next (given green has been drawn)
P(A/B) = 4/9.
P(AB) = P(B) × P(A/B) = 6/10 × 4/9 = 24/90.
Now, when any one of the two situations (red and green or green and red) can happen and both of them are mutually exclusive.
Thus, the required probability will be $= \frac{24}{90} + \frac{24}{90} = \frac{48}{90} = \frac{8}{15}$

Illustration 16: An urn contains 3 white and 2 red balls. Another urn contains 5 white and 3 red balls. A ball is transferred from the first urn to the second and then a ball is drawn from the second urn. Find the probability that the ball is white.

Solution:



Let W be the event that a white ball is transferred from urn I to urn II.
So, P(W) = 3/5
Also, let R be the event that a red ball is transferred from Urn I to Urn II.
Thus, P(R) = 2/5
Let A be the event that a white ball is finally drawn from Urn II
Now, if a white ball is transferred then the total number of balls in Urn II becomes 9. Out of this 9 balls the number of white balls are now 5+1 = 6
Thus, P(White ball from urn II when a white ball was transferred)
= P(A|W) = 6/9 = 2/3
Similarly, if a red ball is transferred then the total number of balls in Urn II becomes 9. Out of this 9 balls the number of white balls are now same as before i.e. 5
Thus, P(White ball drawn from urn II when a red ball was transferred)
= P(A|R) = 5/9
Thus, event a can materialize in the following two mutually exclusive ways.
(a) A red ball was transferred from the first urn to the second and a white ball was drawn from it. Notationally, $R \cap A$
(b) A white ball was transferred from the first urn to the second and a white ball was drawn from it. Notationally, $W \cap A$
Thus, required probability = $P(R \cap A + W \cap A)$
$= P(R \cap A) + P(W \cap A)$
[since, the events are mutually exclusive
$= P(R)\, P(A|R) + P(W)\, P(A|W)$
$= 2/5 \times 5/9 + 3/5 \times 2/3 = 28/45$

EXERCISES

Theoretical

1. Explain the meaning of the terms: (i) experiment (ii) event (iii) simple event and compound event (iv) sample space (v) mutually exclusive events (vi) exhaustive
2. Define union of events, intersection of events and complement of an event.
3. Give the classical definition of probability. What are its limitations?
4. Define independent and mutually exclusive events. Can two events be mutually exclusive and independent simultaneously?
6. State and prove the theorem of total probability for two mutually exclusive events. How is the result modified when the events are not mutually exclusive?
7. Define conditional probability. State and prove the theorem of compound probability.
8. Explain with examples the rules of addition and multiplication in the theory of probability.
9. State and prove Bayes' theorem of conditional probability.
10. State the difference between axiomatic approach and classical definition of probability.

11.Prove that for any two events A and B we have

$P(A+B) = P(A) + P(B) - P(AB)$

12. Define conditional probability.

Problems

13. A card is drawn at random from a well–shuffled pack of cards**.** What is the probability that it is a heart or a queen? [Ans: 4/13]

14. In a single cast with two dice find the chance of throwing 7 (i.e. of throwing two numbers whose sum is 7). [Ans: 1/6]

15. An ordinary die is tossed twice and the difference between die numbers of spots turned up is noted. Find the probability of a difference of 3. [Ans: 1/6]

16. Three balls are drawn at random from a bag containing 6 blue and 4 red balls. What is the chance that two balls are blue and one is red? [Ans: ½]

17. An urn contains 8 white and 3 red balls. If 2 balls are drawn at random, find the probability that (i) both are white, (ii) both are red, (iii) one is of each colour. [Ans: 28/55, 3/55, 24/55]

18. Five persons *A, B, C, D, E* speak at a meeting. What is probability that A speaks immediately before *B?* [Ans: 1/5]

19.The nine digits 1, 2, 3, ..., 9 are arranged in random order form a nine–digit number. Find the probability that 1, 2 and 3 as neighbours in the order mentioned. [Ans: ½, ¼]

20. What is the chance that a leap year selected at random contain 53 Sundays? [Ans: 2/7]

21. Three cards are drawn at random one after another from full pack of Playing Cards. What is the probability that (i) the two are spades and the third is a heart, (ii) two are spades and is a heart? [Ans: 13/850, 39/850]

22. A four–digit number is formed by the digits 1,2.3,4 with no repetition. Find the probability that the number is (i) odd, (ii) divisible by 4. [Ans: ½, ¼]

23. Four dice are thrown. Find the probability that the sum of numbers appearing will be 18. [Ans: 5/81]

24.There are 4 persons in a company. Find the probability that (i) all of them have different birthdays, (ii) at least 2 of them have same birthday, (iii) exactly 2 of them have the same birthday. (Assume 1 year = *365* days). [Ans: 0.984, 0.016, $(6\times364\times363)/365^2$]

25. (i) If P(A) = ¼, P(B)= 2/5, P(A+B) = ½, find P(AB), P(A/B) and P(B/A). [Ans: 3/20, 3/8, 3/5]

(ii) If *P(A)* = ½, *P(B)* = 3/5, P(AB)**=** 1/3, find P(A+B) and *P(AB).* [Ans: 3/20, 3/8, 3/5]

(iii) If *A* and *B* are independent events, and *P(A)* = 2/3, P(B) = *3/5*, find *P(A+B),* $P(\bar{A}/B)$ and $P(\bar{A}B)$. [Ans: 13/15, 1/3, 1/5]

26. A box contains 3 red and 7 white balls. One ball is drawn at random and in its place a ball of the other colour is put in the box. Now one ball is drawn at random from the box. Find the probability that it is red. [Ans: 34/100]

27.There are three urns containing 2 white and 3 black balls, 3 white and 2 black balls and 4 white and 1 black balls respectively. There is equal probability of each urn being chosen. A ball is drawn from an urn chosen at random. Find the probability that a white ball is drawn. [Ans. 3/5]

28.Four cards are drawn at random from a full pack. What is the probability that they belong to different suits? [Ans. 2197/20825]

29.The probability that an entering college student will be a graduate is 0.4. Determine the probability that out of 5 entering students, (i) none, (ii) one, (iii) at least one, will be a graduate. [Ans. (i) 0.07776, (ii) 0.2592, (iii) 0.92224]

30.A card is drawn from a well shuffled pack of 52 cards. Find the probabilities that the card drawn will be (i) red, (ii) spade, (iii) black queen, (iv) king of diamond. [Ans. (i) ½ (ii) ¼ (iii) 1/26 (iv) 1/52]

31.Three coins are tossed in succession. Find out the probabilities of occurrence of (i) two consecutive heads, (ii) two heads, (iii) two heads in the following order, head, tail and head. [Ans. (i) ¼. (ii) 3/8, (iii) 1/8]

32.A coin and a die are thrown simultaneously. What are the probabilities of occurrence of (i) head and even face, (ii) tail and multiple of three. [Ans. (i) ¼, (ii) 1/6]

33.Two letters are drawn at random from the word 'HOME'. Write down the sample space. Now find the probability that

(i) both the letters are vowel.

(ii)at least one is a vowel.

(iii) one of the letters chosen should be M. [Ans. (i) 1/6, (ii)5/6, (iii) ½]

34.Two dice are thrown simultaneously. Find the probability of obtaining (i) double six (ii) no six (iii] at least one six (iv) only one six. [Ans, (i) 1/36, (ii) 25/36 (iii) 11 / 36, (iv) 5/18]

35. Two dice are thrown simultaneously. Find the probability that (i) the sum of the points on two dice is 10 (ii) difference of the points on two dice is 2. [Ans. (i) 1 / 12 (iii) 2/9]

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