Chapter 1

Convergence of Random Variables

1.1 Introduction

In this chapter we deal with the convergence properties of a sequence of random variables. The different types of convergence of random variables dealt with in this chapter are convergence almost surely, convergence in probability, convergence in the r^{th} mean and convergence in distribution. The concept of the convergence of random variables is useful in understanding various features of Statistics like, laws of large numbers, central limit theorem, consistency of estimators and so on. The chapter is prepared based on the discussion made in *"An Introduction to Probability and Statistics"* by Rohatgi and Saleh. The general concept of convergence can be stated as follows, If $X_n = \{X_1, X_2, ...\}$ is a sequence of real numbers we say that $\{X_n\}$ converges to a number X if for all $\epsilon > 0$, there exists N such that $|x_n - x| < \epsilon$ for all $n \geq N$. Based on this idea we can approach different types of convergence of random variables.

1.1.1 Convergence Almost Surely

We say that X_n converges to X almost surely (a.s.) if

$$P\{X_n \to X\} = 1$$

In such a case we write $X_n \to X$ a.s., or $\lim_{n\to\infty} X_n = X$ a.s. or, $X_n \stackrel{a.s.}{\to} X$.
The concept of convergence almost surely is also called convergence almost everywhere. The convergence almost everywhere implies convergence in probability, and hence convergence in distribution as well. This concept of convergence is used for the strong law of large numbers.

1.1.2 Convergence in Probability

We say that X_n converges to X in probability if

$$P\{|X_n - X| \geq \epsilon\} \to 0$$

In such a case we write $X_n \to X$ in probability, or $\lim_{n\to\infty} X_n = X$ in probability or, $X_n \stackrel{P}{\to} X^*$.
The concept of convergence in probability is used in the weak law of large numbers. convergence in probability implies converges in distribution.

1.1.3 Convergence in r-th Mean

We say that X_n converges to X in r^{th} mean if for $r \geq 1$, for all $E|X_n|^r < \infty$, for all n, and

$$E\{(X_n - X)^r\} \to 0$$

This means that the expected value of the difference between X_n and X in the r^{th} power converges to zero. In such a case we write $X_n \to X$ in the r^{th} mean, or $\lim_{n\to\infty} X_n = X$ in the r^{th} mean or, $X_n \xrightarrow{L^r} X$.
Some derived cases for the convergence in r^{th} mean are convergence in mean, obtained by putting $r = 1$ and convergence in mean square obtained by putting $r = 2$.
Now, convergence in r^{th} mean implies convergence in s^{th} mean provided $r > s$. Thus convergence in mean square implies convergence in mean. Also, convergence in r^{th} mean implies convergence in probability and hence convergence in distribution.

1.1.4 Convergence in Distribution

Let F_n be the distribution function of X_n and F^* be the distribution function of X. We say that X_n converges in distribution to X if

$$F_n(x) \to F^*$$

for all x such thatF^* is continuous at x. In such a case we write $X_n \to X$ in distribution, or $\lim_{n\to\infty} X_n = X$ in distribution or, $X_n \xrightarrow{D} X$.
Convergence in distribution is the weakest form of convergence, so it is also called as weak convergence. However, convergence in distribution is implied by all the other type of convergence discussed earlier. Thus, it is the most common and sometimes most useful form of convergence as well as it is used in the central limit theorem and in case of the weak law of large numbers. Convergence in distribution is also sometimes called as convergence in law, here law means probability law i.e. probability distribution, $X_n \xrightarrow{D} X$ is also written as $X_n \xrightarrow{L} X$.

1.2 Some Theorems Related to Convergence in Probability

Theorem 1: Let X_n converges to X in probability and let $g(.)$ be a continuous function defined on $\Re$ then $g(x_n)$converges to $g(x)$ in probability.

Proof: Since $g(x)$ is a continuous function on $\Re$ therefore by the definition of continuity for every $\epsilon > 0 \ \exists \ \delta > 0$ such that,

$$|g(x_n) - g(x)| < \epsilon \ \ if \ \ |x_n - x| < \delta$$

$$i.e. |x_n - x| < \delta \Rightarrow |g(x_n) - g(x)| < \epsilon$$

$$\Rightarrow P[|x_n - x| < \delta] \leq P[|g(x_n) - g(x)| < \epsilon]$$

Since if $A \Rightarrow B$, then $P(A) \leq P(B)$ or

$$\Rightarrow P[|g(x_n) - g(x)| < \epsilon] \geq P[|x_n - x| < \delta]$$

$$\Rightarrow \lim_{n\to\infty} P[|g(x_n) - g(x)| < \epsilon] \geq 1 \ \ since \ \ X_n \xrightarrow{P} X$$

$$\Rightarrow \lim_{n\to\infty} P[|g(x_n) - g(x)| < \epsilon] \to 1$$

$$\Rightarrow g(x_n) \xrightarrow{P} g(x)$$

Theorem 2: If $X_n \xrightarrow{P} X \Leftrightarrow X_n - X \xrightarrow{P} 0$

Proof: Let $X_n - X \xrightarrow{P} 0$, so

$$\begin{aligned} P[|(X_n - X) - 0| > \epsilon] \to 0 \text{ as } n \to \infty \\ P[|X_n - X| > \epsilon] \to 0 \text{ as } n \to \infty \\ \Rightarrow X_n \xrightarrow{P} X \text{ by definition.} \\ \text{Thus, } X_n - X \xrightarrow{P} 0 \Rightarrow X_n \xrightarrow{P} X \end{aligned}$$

Conversely, let $X_n \xrightarrow{P} X$, then from the definition of convergence in probability we have,

$$\begin{aligned} P[|X_n - X| > \epsilon] \to 0 \text{ as } n \to \infty \\ P[|(X_n - X) - 0| > \epsilon] \to 0 \text{ as } n \to \infty \\ \Rightarrow X_n - X \xrightarrow{P} 0 \text{ by definition.} \\ \text{Thus, } X_n \xrightarrow{P} X \Rightarrow X_n - X \xrightarrow{P} 0 \end{aligned}$$

Theorem 3: If $X_n \xrightarrow{P} X \Rightarrow X_n - X_m \xrightarrow{P} 0$ as $n, m \to \infty$.

Proof: Let

$$\begin{aligned} A &= \{|X_n - X_m| < \epsilon\} \\ &= \{|X_n - X + X - X_m| < \epsilon\} \\ &\supseteq \{|X_n - X| + |X - X_m| < \epsilon\} \\ &\supseteq \{|X_n - X| < \frac{\epsilon}{2} \text{ and } |X_m - X| < \frac{\epsilon}{2}\} \\ &= \{|X_n - X| < \frac{\epsilon}{2}\} \cap \{|X_m - X| < \frac{\epsilon}{2}\} \\ &= B \cap C \end{aligned}$$

where $B = \{|X_n - X| < \frac{\epsilon}{2}\}$ and $C = \{|X_m - X| < \frac{\epsilon}{2}\}$
Thus,

$$\begin{aligned} A &\supseteq B \cap C \\ \Rightarrow P(A) &\geq P(B \cap C) \\ \Rightarrow -P(A) &\leq -P(B \cap C) \\ \Rightarrow 1 - P(A) &\leq 1 - P(B \cap C) \\ \Rightarrow P(\bar{A}) &\leq P(\overline{B \cap C}) \\ \Rightarrow P(\bar{A}) &\leq P(\bar{B} \cup \bar{C}) \\ \Rightarrow P(\bar{A}) &\leq P(\bar{B}) + P(\bar{C}) \\ \Rightarrow P(|X_n - X_m| \geq \epsilon) &\leq P(|X_n - X| < \frac{\epsilon}{2}) + P(|X_m - X| < \frac{\epsilon}{2}) \\ \Rightarrow \lim_{n\to\infty} \lim_{m\to\infty} P(|X_n - X_m| \geq \epsilon) &\leq P(|X_n - X| < \frac{\epsilon}{2}) + P(|X_m - X| < \frac{\epsilon}{2}) \\ \Rightarrow \lim_{n\to\infty} \lim_{m\to\infty} P(|X_n - X_m| \geq \epsilon) &\leq 0 \\ \Rightarrow P(|(X_n - X_m) - 0| \geq \epsilon) &\to 0 \text{ as } n, m \to \infty \\ \text{Thus, } X_n - X_m &\xrightarrow{P} 0 \end{aligned}$$

Theorem 4: If $X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y \Rightarrow X_n + Y_n \xrightarrow{P} X + Y$.

Proof: Let us consider the set,

$$\begin{aligned}
\{\left|X_n + Y_n - \overline{X+Y}\right| < \epsilon\} &= \{|X_n + Y_n - X - Y| < \epsilon\} \\
&= \{|X_n - X + Y_n - Y| < \epsilon\} \\
&\supseteq \{|X_n - X| + |Y_n - Y| < \epsilon\} \\
&\supseteq \{|X_n - X| < \frac{\epsilon}{2}\} \cap \{|Y_n - Y| < \frac{\epsilon}{2}\}
\end{aligned}$$

Now considering, $A = \{\left|X_n + Y_n - \overline{X+Y}\right| < \epsilon\}$, $B = \{|X_n - X| < \frac{\epsilon}{2}\}$ and $C = \{|Y_n - Y| < \frac{\epsilon}{2}\}$, we have,

$$\begin{aligned}
A &\supseteq B \cap C \\
\Rightarrow P(\bar{A}) &\leq P(\bar{B}) + P(\bar{C}) \\
\Rightarrow P(\left|X_n + Y_n - \overline{X+Y}\right| \geq \epsilon) &\leq P(|X_n - X| \geq \frac{\epsilon}{2}) + P(|Y_n - Y| \geq \frac{\epsilon}{2}) \\
\Rightarrow P(\left|X_n + Y_n - \overline{X+Y}\right| \geq \epsilon) &\leq 0 \\
\text{Thus, } X_n + Y_n &\xrightarrow{P} X + Y
\end{aligned}$$

<u>Note:</u> Likewise, by taking $A = \{\left|X_n - Y_n - \overline{X-Y}\right| < \epsilon\}$ and proceeding in the similar manner we can show that, $X_n - Y_n \xrightarrow{P} X - Y$. Thus we may write, $X_n \pm Y_n \xrightarrow{P} X \pm Y$ provided $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$.

Theorem 5: If $X_n \xrightarrow{P} X$, k is a constant, then $kX_n \xrightarrow{P} kX$.

Proof: Since,$X_n \xrightarrow{P} X$, so we have,

$$\begin{aligned}
P[|X_n - X| > \epsilon] &\rightarrow 0 \text{ as } n \rightarrow \infty \\
\Rightarrow P[|X_n - X| \leq \epsilon] &\rightarrow 1 \text{ as } n \rightarrow \infty \\
\Rightarrow P[-\epsilon \leq X_n - X \leq \epsilon] &\rightarrow 1 \text{ as } n \rightarrow \infty \\
\Rightarrow P[-k\epsilon \leq kX_n - kX \leq k\epsilon] &\rightarrow 1 \text{ as } n \rightarrow \infty \\
\Rightarrow P[|kX_n - kX| \leq k\epsilon] &\rightarrow 1 \text{ as } n \rightarrow \infty \\
\Rightarrow P[|kX_n - kX| \leq \epsilon_1] &\rightarrow 1 \text{ as } n \rightarrow \infty \\
\text{Thus, } kX_n &\xrightarrow{P} kX
\end{aligned}$$

Theorem 6: If $X_n \xrightarrow{P} k$, where k is a constant, then $X_n^2 \xrightarrow{P} k^2$.

Proof:

$$\begin{aligned}
P[\left|X_n^2 - k^2\right| > \epsilon] &= P[(X_n^2 - k^2) > \epsilon] + P[(X_n^2 - k^2) < -\epsilon] \\
&= P[X_n^2 > k^2 + \epsilon] + P[X_n^2 < k^2 - \epsilon] \\
&= P[X_n > \sqrt{k^2 + \epsilon}] + P[X_n^2 \leq 0] + P[0 < X_n < \sqrt{k^2 - \epsilon}] \\
&\leq P[X_n > k] + P[0 < X_n < \sqrt{k^2 - \epsilon}]
\end{aligned}$$

Both the terms in the right tends to 0 as $n \rightarrow \infty$ So, $X_n^2 \xrightarrow{P} k^2$.

Theorem 7: If $X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y \Rightarrow X_nY_n \xrightarrow{P} XY$, where X and Y are constants.

Proof: We have,

$$X_nY_n = \frac{(X_n+Y_n)^2-(X_n-Y_n)^2}{4}$$

Now, we know that $X_n \pm Y_n \xrightarrow{P} X \pm Y$ provided $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Thus,

$$\frac{(X_n+Y_n)^2-(X_n-Y_n)^2}{4} \xrightarrow{P} \frac{(X+Y)^2-(X-Y)^2}{4} = \frac{4XY}{4} = XY$$

So, $X_nY_n \xrightarrow{P} XY$

Theorem 8: If $X_n \xrightarrow{P} X$, and Y is a random variable, then $X_nY \xrightarrow{P} XY$

Proof: Since Y is a random variable, so we can have, for every $\delta > 0$, there exists $k > 0$, such that, $P[|Y| > k] < \frac{\delta}{2}$. Thus,

$$\begin{aligned} P[|X_nY - XY| > \epsilon] &= P[|X_n - X| > \epsilon, |Y| > k] + P[|X_n - X| > \epsilon, |Y| \le k] \\ &= P[|Y| > k]P[|X_n - X| > \epsilon \mid |Y| > k] + P[|X_n - X| > \epsilon, |Y| \le k] \\ &< \frac{\delta}{2} + P[|X_n - X| > \frac{\epsilon}{k}] \end{aligned}$$

Now, as $n \to \infty$, $P[|X_n - X| > \frac{\epsilon}{k}] \to 0$. So, $P[|X_nY - XY| > \epsilon] \to 0$ as, $n \to \infty$. So, $X_nY \xrightarrow{P} XY$.

Theorem 9: If $X_n \xrightarrow{P} X$, and $Y_n \xrightarrow{P} Y$, then $X_nY_n \xrightarrow{P} XY$

Proof: Since $X_n \xrightarrow{P} X \Rightarrow X_n - X \xrightarrow{P} 0$ and $Y_n \xrightarrow{P} Y \Rightarrow Y_n - Y \xrightarrow{P} 0$. Thus,

$$\begin{aligned} (X_n - X)(Y_n - Y) &\xrightarrow{P} 0 \\ \Rightarrow X_nY_n - XY_n - YX_n - XY &\xrightarrow{P} 0 \\ \Rightarrow X_nY_n - XY &\xrightarrow{P} 0 \\ \Rightarrow X_nY_n &\xrightarrow{P} XY \end{aligned}$$

Since, $XY_n \xrightarrow{P} XY$ and $X_nY \xrightarrow{P} XY$, using the previous theorem.

Theorem 10: If $X_n \xrightarrow{P} X$, then $X_n^2 \xrightarrow{P} X^2$

Proof: Since $X_n \xrightarrow{P} X \Rightarrow X_n - X \xrightarrow{P} 0$. So,

$$\begin{aligned} (X_n - X)(X_n - X) &\xrightarrow{P} 0 \\ \Rightarrow X_n^2 - XX_n - X_nX + X^2 &\xrightarrow{P} 0 \\ \Rightarrow X_n^2 - X^2 &\xrightarrow{P} 0 \end{aligned}$$

Since, $X_n \xrightarrow{P} X \Rightarrow XY_n \xrightarrow{P} XY$ so, putting $X_n = Y_n$, we have $XX_n \xrightarrow{P} X^2$.

1.3 Some Theorems Related to Convergence in Law

Theorem 11: If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{L} X$

Proof: Let us define, the event, $X \leq x'$ such that,

$$\begin{aligned}
(X \leq x') &= (X_n \leq x, X \leq x') \cup (X_n > x, X \leq x') \\
&\subseteq (X_n \leq x) \cup (X_n > x, X \leq x') \\
\text{So, } P(X \leq x') &= P(X_n \leq x) + P(X_n > x, X \leq x') \\
\Rightarrow F(x') &\leq F_n(x) + P(X_n > x, X \leq x') \qquad \cdots (1)
\end{aligned}$$

Now, $X_n \xrightarrow{P} X$ so, $P\{|X_n - X| \geq \epsilon\} \to 0$ as $n \to \infty$
Also, if $x' < x$ then

$$P(X_n > x, X \leq x') = P[|X_n - X| > x - x']$$

Choosing $x - x' = \epsilon$, we would have

$$\begin{aligned}
P(X_n > x, X \leq x') &= P[|X_n - X| > \epsilon] \to 0 \text{ as } n \to \infty \\
\text{Thus, } P(X_n > x, X \leq x') &\to 0 \text{ as } n \to \infty \qquad \cdots (2)
\end{aligned}$$

So, replacing (2) in (1), we have,

$$F(x') \leq F_n(x) \text{ when } n \to \infty \text{ and for } x' < x$$
$$\text{Thus, } F(x') \leq \lim_{n\to\infty} \text{Inf } F_n(x), \quad x' < x \qquad \cdots (3)$$

$$\begin{aligned}
(X_n \leq x) &= (X_n \leq x, X \leq x'') \cup (X_n \leq x, X > x'') \\
&\subseteq (X \leq x'') \cup (X_n \leq x, X > x'') \\
\Rightarrow P(X_n \leq x) &= P(X \leq x'') + P(X_n \leq x, X > x'') \\
\Rightarrow F_n(x) &\leq F(x'') + P(X_n \leq x, X > x'') \qquad \cdots (4)
\end{aligned}$$

Now, $X_n \xrightarrow{P} X$ so, $P\{|X_n - X| > \epsilon\} \to 0$ as $n \to \infty$
Also, if $x'' > x$, then,

$$P(X_n \leq x, X > x'') = P[|X_n - X| > x'' - x]$$

Choosing $x'' - x = \epsilon$, we would have

$$\begin{aligned}
P(X_n \leq x, X > x'') &= P[|X_n - X| > \epsilon] \to 0 \text{ as } n \to \infty \\
\text{Thus, } P(X_n \leq x, X > x'') &\to 0 \text{ as } n \to \infty \qquad \cdots (5)
\end{aligned}$$

So, replacing (5) in (4), we have,

$$F(x'') \geq F_n(x) \text{ when } n \to \infty \text{ and for } x'' > x$$
$$\text{Thus, } \lim_{n\to\infty} \text{Sup } F_n(x) \leq F(x'') \ , x'' > x \qquad \cdots (6)$$

Thus, from (3) and (6) we have,
$F(x') \leq lim\ Inf F_n(x) \leq\ lim\ Sup F_n(x) \leq F(x'')$ for $x' < x < x''$
Now, choosing x as a point at which $F(.)$ is continuous and allowing $x' \to x$ and $x'' \to x$, we have,
$F(x) = \lim_{n\to\infty} F_n(x)$ at all points of continuity of F.

Note: However the converse is not true i.e. $X_n \xrightarrow{L} X$ does not imply $X_n \xrightarrow{P} X$. Let the joint distribution of (X_n, X) is given by,
$P[X_n = 0 \cap X = 0] = P[X_n = 1 \cap X = 1] = 0$ and $P[X_n = 0 \cap X = 1] = P[X_n = 1 \cap X = 0] = \frac{1}{2}$
Now, $F_n(0) = \frac{1}{2}$ and $F_n(1) = 1$ and likewise $F(0) = \frac{1}{2}$ and $F(1) = 1$. Thus,$X_n \xrightarrow{L} X$
Now, $P[|X_n - X| > \frac{1}{2}] = P[X_n = 0, X = 1] + P[X_n = 1, X = 0] = \frac{1}{2} + \frac{1}{2} = 1 \neq 0$
Thus, $X_n \xrightarrow{L} X$ does not imply $X_n \xrightarrow{P} X$.

Theorem 12: If $X_n \xrightarrow{P} C \iff X_n \xrightarrow{L} C$

Proof: Let $X_n \xrightarrow{P} C$
So, we have

$$\begin{aligned}
\lim_{n\to\infty} P[|X_n - C| \geq \epsilon] &= 0 \text{ for } \epsilon > 0 \\
\Rightarrow \lim_{n\to\infty} P[X_n - C \geq \epsilon \text{ or } X_n - C \leq -\epsilon] &= 0 \\
\Rightarrow \lim_{n\to\infty} P[X_n - C \geq \epsilon] + \lim_{n\to\infty} P[X_n - C \leq -\epsilon] &= 0 \qquad \cdots (1)
\end{aligned}$$

From (1) we have,

$$\begin{aligned}
\lim_{n\to\infty} P[X_n - C \geq \epsilon] &= 0 \\
\Rightarrow \lim_{n\to\infty} [1 - P(X_n - C < \epsilon)] &= 0 \\
\Rightarrow \lim_{n\to\infty} P(X_n - C < \epsilon)] &= 1 \\
\Rightarrow \lim_{n\to\infty} P(X_n < \epsilon + C)] &= 1 \\
\Rightarrow F_n(C + \epsilon) &= 1 \qquad \cdots (2)
\end{aligned}$$

Also from (1) we have,

$$\begin{aligned}
\lim_{n\to\infty} P[X_n - C \leq -\epsilon] &= 0 \\
\Rightarrow \lim_{n\to\infty} P[X_n \leq C - \epsilon] &= 0 \\
\Rightarrow F_n(C - \epsilon) &= 0 \qquad \cdots (3)
\end{aligned}$$

Thus, from (2) and (3) we have, $X_n \xrightarrow{L} C$

So, $X_n \xrightarrow{P} C \Rightarrow X_n \xrightarrow{L} C$

Conversely, let $X_n \xrightarrow{L} C$

$$\begin{aligned}
P[|X_n - C| \geq \epsilon] &= P[X_n - C \geq \epsilon \text{ or } X_n - C \leq \epsilon] \\
&= P[X_n - C \geq \epsilon] + P[X_n - C \leq -\epsilon] \\
&= 1 - P[X_n - C < \epsilon] + P[X_n - C \leq -\epsilon] \\
&= 1 - P[X_n < \epsilon + C] + P[X_n \leq C - \epsilon] \\
&= 1 - F_n[\epsilon + C] + F_n[C - \epsilon] \qquad \cdots (4)
\end{aligned}$$

Now, since $X_n \xrightarrow{L} C$, so, we have,

$$\begin{aligned}
\lim_{n\to\infty} F_n(x) &= 0 \text{ if } x < C \\
&= 1 \text{ if } x \geq C
\end{aligned}$$

So, taking limit on both sides of (4), we have,

$$\lim_{n\to\infty} P[|X_n - C| \geq \epsilon] = 1 - \lim_{n\to\infty} F_n(C + \epsilon) + \lim_{n\to\infty} F_n(C - \epsilon) = 0$$

Thus, $X_n \xrightarrow{P} C$ when $X_n \xrightarrow{L} C$
So, we have, $X_n \xrightarrow{P} C \iff X_n \xrightarrow{L} C$

1.4 Some Theorems Related to Convergence in r^{th} Mean

Theorem 13: If X_n converges to X in mean square, then show that $X_n \xrightarrow{P} X$

Proof: The Markov's inequality is given by,

$$\begin{aligned}
P[X \geq a] &\leq \frac{E(X)}{a} \qquad \cdots (1) \\
\text{Given, } X_n &\xrightarrow{2} X \\
\Rightarrow \lim_{n\to\infty} E(X_n - X)^2 &= 0 \\
\text{Replacing } X \text{ by } (X_n - X)^2 \quad &\text{and} \quad a \text{ by } \epsilon^2, \text{ in (1), we have,} \\
P[(X_n - X)^2 \geq \epsilon^2] &\leq \frac{E(X_n - X)^2}{\epsilon^2} \\
\Rightarrow P[(X_n - X) \geq \epsilon] &\leq \frac{E(X_n - X)^2}{\epsilon^2} \\
\Rightarrow \lim_{n\to\infty} P[(X_n - X) \geq \epsilon] &\leq 0
\end{aligned}$$

Thus, $X_n \xrightarrow{P} X$.

Theorem 14: If X_n converges to X in r^{th} mean, then show that $E[|X_n|^r] \to E[|X|^r]$.

Proof: Let $0 < r \leq 1$ then

$$\begin{aligned}
E|X_n|^r &= E|\overline{X_n - X} + X|^r \\
&\leq E|X_n - X|^r + E|X|^r
\end{aligned}$$

Using C.R inequality.[1]

$$\begin{aligned}
\Rightarrow E|X_n|^r - E|X|^r &\leq E|X_n - X|^r \qquad \cdots (1) \\
\text{Interchanging } X_n \text{ and } X\text{, we have,} & \\
\Rightarrow E|X|^r - E|X_n|^r &\leq E|X - X_n|^r \qquad \cdots (2) \\
\text{Combining (1) and (2) we have, } X_n & \\
|E|X_n|^r - E|X|^r| &\leq E|X - X_n|^r
\end{aligned}$$

Taking limit of $n \to \infty$ on both sides and using the fact that $X_n \xrightarrow{r} X$, we have, $E|X_n|^r \to E|X|^r$ as $n \to \infty$ and $0 < r \leq 1$. $\cdots (3)$
Now, let $r \geq 1$. Consider,

$$\begin{aligned}
E|X_n|^r &= E|\overline{X_n - X} + X|^r \\
\Rightarrow E^{\frac{1}{r}}|X_n|^r &\leq E^{\frac{1}{r}}|X_n - X|^r + E^{\frac{1}{r}}|X|^r
\end{aligned}$$

[1] $E|X + Y|^r \leq C_r(E|X|^r + E|Y|^r)$ i.e. $C_r = 1$ for $r \leq 1$ and $= 2^{r-1}$ otherwise.

Using Minkowski's inequality.[2]

$$\Rightarrow E^{\frac{1}{r}}|X_n|^r - E^{\frac{1}{r}}|X|^r \leq E^{\frac{1}{r}}|X_n - X|^r$$

Now taking the limit as $n \to \infty$, in both sides and using the fact that $X_n \xrightarrow{r} X$, we have,

$$\begin{aligned} E^{\frac{1}{r}}|X_n|^r &\to E^{\frac{1}{r}}|X|^r \text{ as } n \to \infty \\ \Rightarrow E|X_n|^r &\to E|X|^r, \quad r \geq 1 \qquad \cdots (4) \end{aligned}$$

Combining (3) and (4), we get the desired result.

Theorem 15: If X_n converges to X in r^{th} mean, and if $r > s$ then show that X_n converges to X in r^{th} mean.

Proof: Let us write $\beta_n = E|X|^n < \infty$.
By Liapounov's inequality, we have for arbitrary chosen k, $2 \leq k < n$

$$\beta_{k-1}^{\frac{1}{k-1}} \leq \beta_k^{\frac{1}{k}}$$

Thus for,$r > s$ (or $\frac{1}{r} < \frac{1}{s}$), we have,

$$\begin{aligned} \{E|X_n - X|^s\}^{\frac{1}{s}} &\leq \{E|X_n - X|^r\}^{\frac{1}{r}} \\ \Rightarrow E|X_n - X|^s &\leq (E|X_n - X|^r)^{\frac{s}{r}} \\ \Rightarrow \lim_{n\to\infty} E|X_n - X|^s &\leq \lim_{n\to\infty} \{E|X_n - X|^r\}^{\frac{s}{r}} \\ \Rightarrow \lim_{n\to\infty} E|X_n - X|^s &\leq 0 \text{ as } \quad X_n \xrightarrow{r} X \end{aligned}$$

Theorem 16: If X_n converges to X in mean square, then show that $E[X_n] \to E[X]$ and $E[X_n^2] \to E[X^2]$

Proof: To prove $E[X_n] \to E[X]$, we proceed as follows:
Since $X_n \xrightarrow{2} X$ so using Theorem 15 we have $X_n \xrightarrow{1} X$

$$\begin{aligned} \text{Thus, } \lim_{n\to\infty} E(X_n - X) &= 0 \\ \Rightarrow \lim_{n\to\infty} E(X_n) &= E(X) \\ \text{Now, } E(X_n^2) &= E[(X_n - X + X)^2] \\ &= E[(X_n - X)^2] + E(X^2) - 2E[(X_n - X)X] \\ &\geq E[(X_n - X)^2] + E(X^2) - 2\sqrt{E[X_n - X]^2 E[X^2]} \\ &\quad \text{using Cauchy-Schwartz inequality } E(|XY|)^2 \leq E(|X|^2)E(|Y|^2) \end{aligned}$$

Taking limit as $n \to \infty$ on both the sides and using the fact that $X_n \xrightarrow{2} X$ we get the desired result.
In addition we get $X_n \to X \Rightarrow V(X_n) \to V(X)$.

[2] $E^{\frac{1}{r}}|X + Y|^r \leq E^{\frac{1}{r}}|X|^r + E^{\frac{1}{r}}|Y|^r$ for $r \geq 1$.

1.5 Some Theorems Related to Convergence in Law

Theorem 17: Let $\{X_n, Y_n\}, n = 1, 2, \cdots$ be a sequence of random variables then
$|X_n - Y_n| \xrightarrow{P} 0$ and $Y_n \xrightarrow{L} Y \Rightarrow X_n \xrightarrow{L} Y$.

Proof: Here we are to show,
$|X_n - Y_n| \xrightarrow{P} 0$ and $Y_n \xrightarrow{L} Y \Rightarrow X_n \xrightarrow{L} Y$
i.e. to show, $X_n - Y_n \xrightarrow{P} 0$ and $Y_n \xrightarrow{L} Y \Rightarrow X_n \xrightarrow{L} Y$.
i.e. to show, $Y_n - X_n \xrightarrow{P} 0$ and $Y_n \xrightarrow{L} Y \Rightarrow X_n \xrightarrow{L} Y$.
Let $\epsilon > 0$ be arbitrary. Let X be a continuity point of the distribution function of Y (i.e. of F_Y. Then,

$$\begin{aligned} P(X_n \leq x) &= P(X_n \leq x + Y_n - Y_n) \\ &= P(Y_n \leq x + (Y_n - X_n)) \\ &= P(Y_n \leq x + (Y_n - X_n), Y_n - X_n \leq \epsilon) \\ &\quad + P(Y_n \leq x + (Y_n - X_n), Y_n - X_n > \epsilon) \\ &\leq P(Y_n \leq x + \epsilon) + P(Y_n - X_n > \epsilon) \\ &\leq P(Y_n \leq x + \epsilon) + P(|Y_n - X_n| \geq \epsilon) \\ \Rightarrow F_{X_n}(x) &\leq F_{Y_n}(x + \epsilon) + P(|Y_n - X_n| \geq \epsilon) \end{aligned}$$

Now, $|X_n - Y_n| \xrightarrow{P} 0$, so,

$$\begin{aligned} \lim_{n\to\infty} F_{X_n}(x) &\leq \lim_{n\to\infty} F_{Y_n}(x + \epsilon) \\ &= F_Y(x + \epsilon) \quad \text{as } Y_n \xrightarrow{L} Y \\ \Rightarrow F_Y(x + \epsilon) &\geq \lim_{n\to\infty} F_{X_n}(x) \\ \Rightarrow F_Y(x + \epsilon) &\geq \lim_{n\to\infty} \text{ Sup } F_{X_n}(x) \end{aligned}$$

$$\cdots (1)$$

Similarly, it can be proved as follows,

$$\begin{aligned} \lim_{n\to\infty} \text{ Inf } F_{X_n}(x) &\geq F_Y(x + \epsilon) \\ \text{Let } P(Y_n \leq x - \epsilon) &= P(Y_n \leq x - \epsilon + X_n - X_n) \\ &= P(X_n \leq x + (X_n - Y_n) - \epsilon) \\ &= P(X_n \leq x + (X_n - Y_n) - \epsilon, X_n - Y_n \leq \epsilon) \\ &\quad P(X_n \leq x + (X_n - Y_n) - \epsilon, X_n - Y_n > \epsilon) \\ &\leq P(X_n \leq x) + P(X_n - Y_n \geq \epsilon) \\ \Rightarrow F_{Y_n}(x - \epsilon) &\leq F_{X_n}(x) + P(X_n - Y_n \geq \epsilon) \\ \lim_{n\to\infty} F_{Y_n}(x - \epsilon) &\leq \lim_{n\to\infty} F_{X_n}(x) + \lim_{n\to\infty} P(|X_n - Y_n| \geq \epsilon) \\ \Rightarrow F_{Y_n}(x - \epsilon) &\leq \lim_{n\to\infty} F_{X_n}(x) \text{ Since, } |X_n - Y_n| \xrightarrow{P} 0 \text{ and } Y_n \xrightarrow{L} Y \\ \Rightarrow F_{Y_n}(x - \epsilon) \leq \lim_{n\to\infty} \text{ Inf } F_{X_n}(x) &\leq \lim_{n\to\infty} \text{ Sup } F_{X_n}(x) \leq F_{Y_n}(x + \epsilon) \\ \text{Letting } \epsilon \to 0, \text{ we have,} & \\ \Rightarrow F_{Y_n}(x) \leq \lim_{n\to\infty} \text{ Inf } F_{X_n}(x) &\leq \lim_{n\to\infty} \text{ Sup } F_{X_n}(x) \leq F_{Y_n}(x) \\ \Rightarrow \lim_{n\to\infty} F_{X_n}(x) &\leq F_{Y_n}(x) \\ \Rightarrow X_n \xrightarrow{L} Y & \end{aligned}$$

Thus, $|X_n - Y_n| \xrightarrow{P} 0$ and $Y_n \xrightarrow{L} Y \Rightarrow X_n \xrightarrow{L} Y$

Theorem 18: Let $\{X_n, Y_n\}, n = 1, 2, \cdots$ be a sequence of pairs of random variables and C is a constant, then
(a) $X_n \xrightarrow{L} X$, $Y_n \xrightarrow{P} C \Rightarrow X_n + Y_n \xrightarrow{L} X \pm C$.
(b) $X_n \xrightarrow{L} X$, $Y_n \xrightarrow{P} C \Rightarrow X_n Y_n \xrightarrow{L} CX$, $C \neq 0$.
(c) $X_n \xrightarrow{L} X$, $Y_n \xrightarrow{P} 0 \Rightarrow X_n Y_n \xrightarrow{P} 0$.
(d) $X_n \xrightarrow{L} X$, $Y_n \xrightarrow{P} C \Rightarrow \frac{X_n}{Y_n} \xrightarrow{L} \frac{X}{C}, C \neq 0$.

Proof: We have from the previous theorem that,
$|X_n - Y_n| \xrightarrow{P} 0$ and $Y_n \xrightarrow{L} Y \Rightarrow X_n \xrightarrow{L} Y$.
(a) Since, $X_n \xrightarrow{L} X$, therefore $X_n \pm C \xrightarrow{L} X \pm C$

$$\begin{aligned}
\text{Also, } Y_n \xrightarrow{P} C &\Rightarrow Y_n - C \xrightarrow{P} 0 \\
\Rightarrow Y_n + X_n - (X_n - C) \xrightarrow{P} 0 &\Rightarrow Y_n + X_n \xrightarrow{P} (X_n - C) \\
\Rightarrow Y_n + X_n &\xrightarrow{L} X + C \\
\text{Also, } Y_n \xrightarrow{P} C &\Rightarrow Y_n - C \xrightarrow{P} C \\
\text{So, } X_n - C - (X_n - Y_n) \xrightarrow{P} 0 &\rightarrow X_n - Y_n \xrightarrow{P} X_n - C \\
\text{So, } X_n - Y_n \xrightarrow{L} X - C &
\end{aligned}$$

(b) When $C \neq 0$
$X_n \xrightarrow{L} X$, $Y_n \xrightarrow{P} C \Rightarrow X_n Y_n \xrightarrow{L} CX$,
Consider,

$$XY_n - CX_n = X_n(Y_n - C)$$

Since, $X_n \xrightarrow{L} X$ and $Y_n \xrightarrow{P} C$ or $Y_n - C \xrightarrow{P} 0 \Rightarrow CX_n \xrightarrow{L} CX$
It follows that,

$$\begin{aligned}
X_n(Y_n - C) &\xrightarrow{P} 0 \\
\Rightarrow X_n Y_n - CX_n &\xrightarrow{P} 0 \\
\text{Using} |X_n Y_n| \xrightarrow{P} 0, Y_n \xrightarrow{L} Y \Rightarrow X_n \xrightarrow{L} Y & \\
\Rightarrow X_n Y_n \xrightarrow{L} CX &
\end{aligned}$$

(c)$X_n \xrightarrow{L} X$ and $Y_n \xrightarrow{P} 0 \Rightarrow X_n Y_n \xrightarrow{P} 0$
We have for any fixed number $k > 0$,

$$\begin{aligned}
P(|X_n Y_n| \geq \epsilon) &= P(|X_n Y_n| \geq \epsilon, |Y_n| \leq \frac{\epsilon}{k}) + P(|X_n Y_n| \geq \epsilon, |Y_n| > \frac{\epsilon}{k}) \\
&\leq P(|X_n| \geq k) + P(|Y_n| > \frac{\epsilon}{k})
\end{aligned}$$

Since, $Y_n \xrightarrow{P} 0$, and $X_n \xrightarrow{L} X$, it follows for any fixed $k > 0$

$$\begin{aligned}
\lim_{n\to\infty} P(|X_n Y_n| \geq \epsilon) &\leq P(|X_n| \geq k) \\
&= 1 - P(|X_n| < k)
\end{aligned}$$

Since, k is arbitrary we can make $P(|X| \geq k)$ as small as we decrease by choosing k large.

$$\begin{aligned}\lim_{n\to\infty} P(|X_nY_n| \geq \epsilon) &\leq 0\\ \Rightarrow |X_nY_n| &\xrightarrow{P} 0\end{aligned}$$

(d)Now, $X_n \xrightarrow{L} X$ and $Y_n^{-1} \xrightarrow{P} C^{-1}$

$$\begin{aligned}\Rightarrow X_nY_n^{-1} &\xrightarrow{L} XC^{-1}\\ \Rightarrow \frac{X_n}{Y_n} &\xrightarrow{L} \frac{X}{C}\end{aligned}$$

1.6 Solved Illustrations

Illustration 1: Let $\{X_n\}$ be a sequence of random variables such that,

$$\begin{aligned}P[X_n = 1] &= \frac{1}{n} \text{ and}\\ P[X_n = 0] &= 1 - \frac{1}{n}\end{aligned}$$

Then show that X_n converges in mean square to X, where X is a random variable degenerate at zero. Also show that X_n converges to 0 in probability.

Solution: Here we have,

$$\begin{aligned}E[X_n^2] &= 1^2 \times P[X_n = 1] + 0^2 \times P[X_n = 0]\\ &= 1 \times \frac{1}{n} = \frac{1}{n}\end{aligned}$$

So,

$$\begin{aligned}E[X_n^2] &\to 0 \text{ as } n \to \infty\\ E[|X_n - 0|^2] &\to 0 \text{ as } n \to \infty\\ \Rightarrow X_n &\xrightarrow{2} 0 \Rightarrow X_n \xrightarrow{2} X \text{ where, } X = 0\end{aligned}$$

Thus, the random variable X is degenerate at 0. Now,

$$\begin{aligned}P[X_n = 1] &= \frac{1}{n} \text{ and}\\ P[X_n = 0] &= 1 - \frac{1}{n} \text{ implies}\\ P[|X_n| > 0] &= \frac{1}{n}\\ \Rightarrow P[|X_n| > \epsilon] &= \frac{1}{n} \text{ if } 0 < \epsilon < 1\\ &= 0 \text{ if } \epsilon \geq 1\end{aligned}$$

Thus;

$$\begin{aligned}\lim_{n\to\infty} P[|X_n| > \epsilon] &= 0\\ \Rightarrow \lim_{n\to\infty} P[|X_n - 0| > \epsilon] &= 0\\ \Rightarrow X_n \xrightarrow{P} 0\end{aligned}$$

Illustration 2: Let $X_1, X_2, \cdots, X_n$ be standardized random variables with $E(X_i^4) < \infty$. Find the limiting distribution of

$$Z_n = \frac{\sqrt{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n})}{X_1^2 + X_2^2 + ... + X_n^2}$$

Solution : Let $Y_j = X_{2j-1}X_{2j}$, accordingly we have,
$E(Y_j) = 0$ and $V(Y_j) = 1$ and so,
$\sum_{j=1}^{n} Y_j \sim N(0, n) \Rightarrow \sum_{j=1}^{n} \frac{Y_j}{\sqrt{n}} \sim N(0, 1)$
Thus we have, $\frac{X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}}{\sqrt{n}} \sim N(0, 1)$
$\Rightarrow \frac{X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n}}{\sqrt{n}} \xrightarrow{L} N(0, 1)$ $\cdots (1)$
Now,

$$\begin{aligned} E(X_i^2) &= 1 \\ \Rightarrow X_1^2 + X_2^2 + ... + X_n^2 &\xrightarrow{P} n \\ \Rightarrow \frac{X_1^2 + X_2^2 + ... + X_n^2}{n} &\xrightarrow{P} 1 \qquad \cdots (2) \end{aligned}$$

We know that, if

$$\begin{aligned} X_n \xrightarrow{L} X \quad &\text{and} \quad Y_n \xrightarrow{P} C, (C \neq 0) \\ \text{Then, } \frac{X_n}{Y_n} &\xrightarrow{L} \frac{X}{C} \end{aligned}$$

Thus, from (1) and (2) we have,

$$Z_n = \frac{\sqrt{n}(X_1X_2 + X_3X_4 + ... + X_{2n-1}X_{2n})}{X_1^2 + X_2^2 + ... + X_n^2} \sim N(0, 1)$$

Illustration 3: Let $X_1, X_2, \cdots, X_n$ be i.i.d N(0, 1) variates, then find the limiting distribution of

$$\frac{\sqrt{n}(X_1 + X_2 + ... + X_n)}{X_1^2 + X_2^2 + ... + X_n^2}$$

Solution : Here we have, $X_i \sim N(0, 1)$ for all i

$$\begin{aligned} \text{Thus we have, } X_1 + X_2 + ... + X_n &\sim N(0, n) \\ \Rightarrow \frac{X_1 + X_2 + ... + X_n}{\sqrt{n}} &\sim N(0, 1) \end{aligned}$$

Thus, $\frac{X_1 + X_2 + ... + X_n}{\sqrt{n}} \xrightarrow{L} N(0, 1)$ $\cdots (1)$

Again,

$$\begin{aligned} X_1^2 + X_2^2 + ... + X_n^2 &\sim \chi_n^2 \\ \Rightarrow E(X_1^2 + X_2^2 + ... + X_n^2) &= n \\ \Rightarrow X_1^2 + X_2^2 + ... + X_n^2 &\xrightarrow{P} n \qquad \cdots (2) \\ \text{Again, } X_n \xrightarrow{L} X \quad &\text{and} \quad Y_n \xrightarrow{P} C, (C \neq 0) \\ \text{Then, } \frac{X_n}{Y_n} &\xrightarrow{L} \frac{X}{C} \end{aligned}$$

Thus, from (1) and (2) we have,

$$\frac{\sqrt{n}(X_1 + X_2 + ... + X_n)}{X_1^2 + X_2^2 + ... + X_n^2} \sim N(0, 1)$$

Illustration 4: Let $X_1, X_2, \cdots, X_n$ be i.i.d N(0, 1) variates, then find the limiting distribution of

$$Z_n = \frac{U_n}{V_n}, \text{ where}$$

$$U_n = \left(\frac{X_1}{X_2} + \frac{X_3}{X_4} + ... + \frac{X_{2n-1}}{X_{2n}}\right)$$
$$\text{and } V_n = X_1^2 + X_2^2 + ... + X_n^2$$

Solution : Here we have, $X_i \sim N(0, 1)$ for all i
Thus,

$$\begin{aligned}
Y_j = \frac{X_{2j-1}}{X_{2j}} &\sim C(1, 0) \\
\Rightarrow \sum_{j=1}^{n} \frac{Y_j}{n} &\sim C(1, 0) \\
\Rightarrow \sum_{j=1}^{n} \frac{Y_j}{n} &\sim C(1, 0) \text{ as } n \to \infty \\
\Rightarrow \sum_{j=1}^{n} \frac{Y_j}{n} &\xrightarrow{L} C(1, 0) \qquad \cdots (1)
\end{aligned}$$

Also we have,

$$\begin{aligned}
V_n = X_1^2 + X_2^2 + ... + X_n^2 &\sim \chi_n^2 \\
\Rightarrow E(V_n) &= n \\
\Rightarrow V_n &\xrightarrow{P} n \\
\Rightarrow \frac{V_n}{n} &\xrightarrow{P} 1 \qquad \cdots (2) \\
\text{Again, } X_n \xrightarrow{L} X \quad &\text{and} \quad Y_n \xrightarrow{P} C, (C \neq 0) \\
\text{Then, } \frac{X_n}{Y_n} &\xrightarrow{L} \frac{X}{C}
\end{aligned}$$

Thus, from (1) and (2) we have,

$$Z_n = \frac{U_n}{V_n} \sim C(1, 0)$$

Illustration 5: Let X be a degenerate random variable degenerate at the point $X = \mu$. Let $X_1, X_2, ..., X_n$ be n i.i.d random variables, with mean μ and variance σ^2. Z_n is defined as $Z_n = \frac{X_1 + X_2 + ... + X_n}{n}$. Show that, $Z_n \xrightarrow{P} X$.

Solution : Here we have, X as a degenerate random variable degenerate at the point $X = \mu$.

So,

$$\begin{aligned}
P(X=\mu) &= 1 \\
\text{Also } E(X_i) &= \mu \\
\text{and } V(X_i) &= \sigma^2 \\
\Rightarrow E(Z_n) &= E\left(\frac{X_1+X_2+...+X_n}{n}\right) = \mu \\
\text{and } V(Z_n) &= \frac{\sigma^2}{n}
\end{aligned}$$

Then by Chebyshev's Inequality, we have

$$\begin{aligned}
P(|Z_n-\mu|>\epsilon) &\leq \frac{\sigma^2}{n\epsilon^2} \\
\Rightarrow \lim_{n\to\infty} P(|Z_n-\mu|>\epsilon) &= 0 \\
\Rightarrow \lim_{n\to\infty} P(|Z_n-X|>\epsilon) &= 0
\end{aligned}$$

Hence, $Z_n \xrightarrow{P} X$ using the degenerate nature of X at μ.

Illustration 6: Let $X_1, X_2, \cdots$ be a sequence of random variables with corresponding distribution function given by $F_n(x) = 0$ if $x < -n, = (x+n)/2n$ if $-n \leq x < n$, and $= 1$ if $x \geq n$. does F_n converge to a distribution function?

Solution: Here we have,

$$\begin{aligned}
F_n(x) &= 0 \text{ if } x < -n \\
&= \frac{x+n}{2n} \text{ if } -n \leq x < n \\
&= 1 \text{ if } x \geq n \\
\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\frac{x+n}{2n} &= \lim_{n\to\infty}\frac{1+\frac{x}{n}}{2} = \frac{1}{2} \\
\text{Thus, } F(-\infty) &= 0 \\
F(\infty) &= 1 \\
\text{and,} F(x-) \leq \ F(x) &\leq F(x+)
\end{aligned}$$

with jump points existing at $-n$ and n and continuous at the jump points thus F(x) is a distribution function as well.

Illustration 7: Let $X_1, X_2, \cdots$ be a sequence of random variables which are iid $U(0,1)$. Let $X_{(1)} = min(X_1, X_2, \cdots, X_n)$ and consider a sequence $Y_n = nX_{(1)}$. Does Y_n converge in distribution to some random variable Y. If so, what is the distribution function of Y?

Solution: Here we have, $X_1, X_2, \cdots, \sim U(0,\theta)$. So, we have

$f(x_i) = \frac{1}{\theta}$ and so, $F(x_i) = 1 - \frac{x}{\theta}$

$$
\begin{aligned}
f_1(x) &= n(1 - F(x))^{n-1} f(x) \\
&= n\left(1 - \frac{x}{\theta}\right)^{(n-1)} \frac{1}{\theta} \\
\text{Now, let } Y_n = nX_{(1)} &\quad \text{so} \quad |J| = \frac{1}{n} \\
\text{So, } g(y) &= \left(1 - \frac{y}{n\theta}\right)^{(n-1)} \frac{1}{\theta} \\
\Rightarrow G(y) &= \frac{1}{\theta}\int_0^y \left(1 - \frac{y}{n\theta}\right)^{(n-1)} dy \\
&= 1 - \left(1 - \frac{y}{n\theta}\right)^n \\
\text{Now, } \lim_{n\to\infty} G(y) &= \lim_{n\to\infty} 1 - \left(1 - \frac{y}{n\theta}\right)^n \\
&= 1 - e^{-\frac{y}{\theta}}, \text{ which is the DF of a exponential variate.}
\end{aligned}
$$

Thus, Y converges to an exponential variate with mean $\frac{1}{\theta}$ in law.

Illustration 8: Let $X_1, X_2, \cdots$ be a sequence of iid random variables with PDF $f(x) = e^{-x+\theta}$ if $x \geq \theta$, and $= 0$ if $x < \theta$. Show that $\overline{X}_n \xrightarrow{P} 1 + \theta$.

Solution: Here we have,

$$
\begin{aligned}
f(x) &= e^{-x+\theta}, \quad x \geq \theta \\
&= 0, \quad x < \theta \\
\text{Also, } E(\overline{X}_n) &= E\left(\frac{\sum X_i}{n}\right) = \frac{1}{n}\sum E(X_i) \qquad \cdots(1) \\
E(X_i) &= \int_\theta^\infty x e^{-x+\theta} dx \\
\text{Now, putting } x - \theta = y & \\
&= \int_0^\infty (y+\theta) e^{-y} dy \\
&= \int_0^\infty y e^{-y} dy + \int_0^\infty \theta e^{-y} dy \\
&= 1 + \theta \qquad \cdots(2) \\
E(X_1 + X_2 + ... + X_n) &= nE(X_i) = n(1+\theta) \qquad \cdots(3) \\
\Rightarrow E(\overline{X}_n) &= 1 + \theta \text{ replacing (3) in (1).} \\
\Rightarrow \lim_{n\to\infty} E(\overline{X}_n) &= 1 + \theta \\
\Rightarrow \overline{X}_n &\xrightarrow{1} 1 + \theta \\
\Rightarrow \overline{X}_n &\xrightarrow{P} 1 + \theta
\end{aligned}
$$

Illustration 9: Let $X_1, X_2, \cdots$ be a sequence of iid random variables with PDF $f(x) = \theta e^{-\theta x}$ if $x \geq 0$, and $= 0$ if $x < 0$. Show that $minX_n \xrightarrow{1} 0$.

Solution: Here we have,

$$\begin{aligned}
f(x) &= \theta e^{-\theta x}, \quad x \geq 0 \\
F(x) &= \int_0^x \theta e^{-\theta x} dx = 1 - e^{-\theta x} \\
\text{Let, } Y = Min(X_1, X_2, ..., X_n) & \\
\text{So, } g(y) &= n[1 - F(y)]^{n-1} f(y) \\
&= n\theta e^{-n\theta y} \\
\Rightarrow E(Y) &= \frac{1}{n\theta} \\
\Rightarrow \lim_{n\to\infty} E(Y) &= 0 \\
\Rightarrow Y \xrightarrow{1} 0 &
\end{aligned}$$

Thus, $minX_n \xrightarrow{1} 0$.

Illustration 10: If X_n converges in mean square to zero in such a way that $\sum_{n=1}^{\infty} E(X_n^2) < \infty$, then it follows that this converges almost surely to zero.

Solution: Here we have,

$$\begin{aligned}
\sum_{n=1}^{\infty} E(X_n^2) &< \infty \\
\Rightarrow E(\sum_{n=1}^{\infty} X_n^2) &< \infty \\
\text{Now, if } E(X) < \infty \text{ then} \quad X & \quad \text{is a finite random variable with probability 1.} \\
\text{Thus, } E(\sum_{n=1}^{\infty} X_n^2) < \infty &\Rightarrow P(\sum_{n=1}^{\infty} X_n^2 < \infty) = 1
\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} X_n^2$ is a convergent series, and for a convergent series we have the n^{th} term in limit tends to 0.

$$\begin{aligned}
\text{Hence, } P(\lim_{n\to\infty} X_n^2 = 0) &= 1 \\
\Rightarrow P(\lim_{n\to\infty} X_n = 0) &= 1 \\
\text{Thus, } X_n \xrightarrow{a.s.} 0 &
\end{aligned}$$

Exercise

1. Explain the different types of convergence of random variables with the help of illustrations.

2. Let $\{X_n\}$ be a sequence of independent random variables such that $P(X_n = 0) = 1 - \frac{1}{n^2}$ and n = 1, 2, ... Then show that X_n does not converge to zero in mean square.

3. Let $\{X_n\}$ be a sequence of independent random variables such that $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Then $X_n \xrightarrow{a.s.} 0$ if $\sum p_n$ is convergent.

4. Let $X_1, X_2, \cdots$ be a sequence of iid random variables with PDF $f(x) = e^{-x+\theta}$ if $x \geq \theta$, and $= 0$ if $x < \theta$. Show that min $(X_1, X_2, \cdots, X_n)$ converges to θ in probability.

5. Let $X_1, X_2, \cdots$ be a sequence of iid random variables with PDF U(0, θ), then show that max $(X_1, X_2, \cdots, X_n)$ converges to θ in probability.

6. Let $\{X_n\}$ be a sequence of independent geometric random variables with parameter $\frac{m}{n}$ where $n > m > 0$. Let us define $Z_n = X_n/n$. Show that $Z_n \xrightarrow{L} G(1, 1/m)$.

7. Let $\{X_n\}$ be a sequence of independent random variables such that $X_n \xrightarrow{L} X$. Let a_n be a sequence of positive constants such that $a_n \to \infty$ as $n \to \infty$. Show that $a_n^{-1} X_n \xrightarrow{P} 0$.

8. Show with the help of an example that convergence in distribution does not imply convergence in probability.