Multivariate Random Variable

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In probability theory, a multivariate random variable or random vector is a list or vector of random variables each of whose value is unknown. The individual random variables combine to form a random vector and are grouped together as they are all part of a single statistical system - often they represent different properties of an individual statistical unit. For example, while a given person has a specific age, height and weight, the representation of these features of an unspecified person from within a group would be a random vector. Normally each element of a random vector is a real number.

More formally, a multivariate random variable is a column vector **X** = ($X_1, X_2, \ldots, X_p$)′ or $\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}$

(or its transpose, which is a row vector) whose components are scalar-valued random variables on the same probability space as each other, $(\Omega, A.P[.])$, where Ω is the sample space, A is the sigma-algebra (the collection of all events), and P[.] is the probability measure (a function returning each event's probability).

Every random vector gives rise to a probability measure on $\mathbf{R}^n$ is known as the joint probability distribution, the joint distribution, or the multivariate distribution of the random vector.

The distributions of each of the component random variables X_i are called marginal distributions. The conditional probability distribution of X_i given X_j is the probability distribution of X_i when X_j is known to be of a particular value.

The mathematical expectation of the random vector ***X*** is a vector of the expected value of the individual variables X_i ($i = 1, 2, \ldots, p$). More precisely,

$$E(\mathbf{X}) = E\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} = \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_p) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_{2.} \\ \vdots \\ \mu_p \end{pmatrix} = \boldsymbol{\mu}$$

Where, $\mu_i = \int_{-\infty}^{\infty} x_i f(x_i) dx_i$ in case X_i is a continuous random variable

$= \sum x_i p_i$ in case X_i is a discrete random variable

The covariance matrix (also called second central moment or variance-covariance matrix of a random vector is of the form $p \times 1$ is an $p \times p$ matrix whose $(i, j)^{th}$ element is the covariance between the i^{th} and the j^{th} random variables. The covariance matrix is the expected value, element by element, of the $p \times p$ matrix computed as E[(**X** - E(**X**)) (**X** - E(**X**))′], where the superscript ′ refers to the transpose of the indicated vector.

So,

$$\Sigma = E[(\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))'] = E[(\mathbf{X} - \mu)(\mathbf{X} - \mu)']$$

$$= E\begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_{2.} \\ . \\ . \\ . \\ X_p - \mu_p \end{pmatrix}\left(X_1 - \mu_1, X_2 - \mu_2 ... X_p - \mu_p\right)$$

$$= \begin{bmatrix} E(X_1 - \mu_1)^2 & E(X_1 - \mu_1)(X_2 - \mu_2) & ... & E(X_1 - \mu_1)(X_p - \mu_p) \\ E(X_2 - \mu_2)(X_1 - \mu_1) & E(X_2 - \mu_2)^2 & ... & E(X_2 - \mu_2)(X_p - \mu_p) \\ ... & ... & ... & ... \\ E(X_p - \mu_p)(X_1 - \mu_1) & E(X_p - \mu_p)(X_2 - \mu_2) & ... & E(X_p - \mu_p)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & ... & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & ... & \sigma_{2p} \\ ... & ... & ... & ... \\ \sigma_{p1} & \sigma_{p2} & ... & \sigma_p^2 \end{bmatrix} = \mathrm{Cov}(\mathbf{X})$$

Where, $\sigma_i^2 = E(X_i - E(X_i))^2 = E(X_i - \mu_i)^2 = \int (x_i - \mu_i)^2 f_i(x_i) dx_i$ and

$$\sigma_{ij} = E(X_i - E(X_i))(X_j - E(X_j)) = \int (x_i - \mu_i)(x_j - \mu_j) f_{ij}(x_i, x_j) dx_i dx_j$$

Multivariate Density Functions, Marginal and Conditional Densities

Let **X** = $\begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ / \\ X_p \end{pmatrix}$ be a random vector

Let $f(x_1, x_2, \ldots, x_p)$ be the density function of the random vector **X**. So, we have,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f(x_1, x_2, ..., x_p) dx_1 dx_2 ... dx_p = 1$$

Now, the random vector **X** be subdivided into two sub-vectors $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ where we have

$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ X_k \\ X_{k+1} \\ X_{k+2} \\ \cdots \\ X_p \end{bmatrix}$ which is divided into two sub vectors $\mathbf{X}^{(1)} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}$ and $\mathbf{X}^{(2)} = \begin{pmatrix} X_{k+1} \\ X_{k+2} \\ \vdots \\ X_p \end{pmatrix}$ where $\mathbf{X}^{(1)}$ is of order k × 1 and $\mathbf{X}^{(2)}$ is of order (p-k) × 1

Then the marginal distribution of $\mathbf{X}^{(1)}$ is given by,

$g_1(x_1, x_2, ..., x_k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} f(x_1, x_2, ..., x_p)dx_{k+1}dx_{k+2}...dx_p$ i.e. the joint density function of all the random variables in the random vector **X** is integrated in the entire range for the random variables classified under $\mathbf{X}^{(2)}$ to get the marginal distribution of $\mathbf{X}^{(1)}$

Similarly, the marginal distribution of $\mathbf{X}^{(2)}$ is given by,

$$g_2(x_{k+1}, x_{k+2}, ..., x_p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} f(x_1, x_2, ..., x_p)dx_1dx_2...dx_k$$

The conditional density function of $\mathbf{X}^{(1)}$ when the other vector $\mathbf{X}^{(2)}$ is already observed as $\mathbf{x}^{(2)}$, is given by,

$$g_3\left(X^{(1)} \mid X^{(2)} = x^{(2)}\right) = \frac{f(x_1, x_2, ..., x_p)}{g_2(x_{k+1}, x_{k+2}, ..., x_p)} = \frac{f(x_1, x_2, ..., x_p)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} f(x_1, x_2, ..., x_p)dx_1dx_2...dx_k}$$

Likewise, we can have the conditional density function of $\mathbf{X}^{(2)}$ when the other vector $\mathbf{X}^{(1)}$ is already observed as $\mathbf{x}^{(1)}$ as,

$$g_4\left(X^{(2)} \mid X^{(1)} = x^{(1)}\right) = \frac{f(x_1, x_2, ..., x_p)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} f(x_1, x_2, ..., x_p)dx_{k+1}dx_{k+2}...dx_p}$$

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